

CONTINUOUS BILINEAR MAPS ON BANACH $\star$ -ALGEBRAS

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ABSTRACT. Let A be a unital Banach $\star$ -algebra with unity 1, X be a Banach space and $\phi : A \times A \rightarrow X$ be a continuous bilinear map. We characterize the structure of ϕ where it satisfies any of the following properties:

$$a, b \in A, \quad ab^\star = z \ (a^\star b = z) \Rightarrow \phi(a, b^\star) = \phi(z, 1) \ (\phi(a^\star, b) = \phi(z, 1));$$

$$a, b \in A, \quad ab^\star = z \ (a^\star b = z) \Rightarrow \phi(a, b^\star) = \phi(1, z) \ (\phi(a^\star, b) = \phi(1, z)),$$

where $z \in A$ is fixed.

1. INTRODUCTION

In recent years, several authors studied the linear (additive) maps that behave like homomorphisms, derivations or right (left) centralizers when acting on special products (for instance, see [1, 3, 4, 6, 7, 8, 9, 10, 16, 17] and the references therein). The above questions and the question of characterizing linear maps that preserve special products on algebras can be solved by considering bilinear maps that preserve certain product properties. Motivated by these reasons, Brešar et al. [5] introduced the concept of zero product (resp., Jordan product, Lie product) determined algebras. In the continuation of this discussion, the problem of characterizing bilinear maps at specific products was considered. We refer the reader to [2, 11, 12, 13, 14, 15, 19] and references therein for results concerning characterizing bilinear maps through special products. With the idea of the above, in this article we will characterize continuous bilinear maps on Banach $\star$ -algebras through special products based on the action of the involution. Our results can be useful in studying the structure of Banach $\star$ -algebras. Proving our main result is also technical and it is based on complex analysis. In the second section, some preliminaries and necessary tools are presented. The third section contains the main results of the article.

2. PRELIMINAIRES

Let A be a Banach $\star$ -algebra. In this article, we will consider the following sets, which are defined based on specific multiplications.

$$S_A^{r\star}(z) = \{(a, b) \in A \times A : ab^\star = z\},$$

$$S_A^{l\star}(z) = \{(a, b) \in A \times A : a^\star b = z\},$$

where $z \in A$ is a fixed point.

In order to prove our results we need the following lemmas from the complex analysis, see [18].

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Lemma 2.1. *If series $\sum_{n=0}^{\infty} a_n z^n$ converges for all real values of z , then it converges for all $z \in \mathbb{C}$.*

Lemma 2.2. *Suppose that*

- (a) *a function f is analytic throughout a domain D ;*
- (b) *$f(z) = 0$ at each point z of a domain or line segment contained in D .*

Then $f(z) \equiv 0$ in D ; that is, $f(z)$ is identically equal to zero throughout D .

3. CONTINUOUS BILINEAR MAPS OF BANACH $\star$ -ALGEBRAS THROUGH SPECIAL PRODUCTS

In this section we will give our main results. Through this section A is a unital Banach $\star$ -algebra with unity 1, $z \in A$ is a fixed, and X is a Banach space.

Theorem 3.1. *Let X be a Banach space and let $\phi : A \times A \rightarrow X$ be a continuous bilinear map with the property that*

$$\phi(a, b^{\star}) = \phi(z, 1) \text{ for all } (a, b) \in S_A^{r\star}(z) \text{ [} (a, b) \in S_A^{l\star}(z) \text{]}.$$

Then

$$\phi(za, a) = \phi(za^2, 1) \text{ and } \phi(za, 1) = \phi(z, a), \quad a \in A$$

and there exists a continuous linear map $\Phi : A \rightarrow X$ such that

$$\phi(za, b) + \phi(zb, a) = \Phi(a \circ b), \quad a, b \in A.$$

Proof. Let $a \in A_{sa}$ and $t \in \mathbb{R}$. Since $(z \exp(ta), \exp(-ta)) \in S_A^{r\star}(z)$, we deduce that

$$\begin{aligned} \phi(z, 1) &= \phi(z \exp(ta), (\exp(-ta))^{\star}) \\ &= \phi\left(z \exp(ta), \left(\sum_{m=0}^{\infty} \frac{(-t)^m a^m}{m!}\right)^{\star}\right) \\ &= \phi\left(\exp(ta), \left(\sum_{m=0}^{\infty} \frac{(-1)^m t^m a^m}{m!}\right)\right) \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m t^m}{m!} \phi(z \exp(ta), a^m) \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m t^m}{m!} \phi\left(\sum_{n=0}^{\infty} \frac{t^n z a^n}{n!}, a^m\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^m t^{m+n}}{m!n!} \phi(z a^n, a^m) \\ &= \phi(z, 1) + \sum_{k=1}^{\infty} t^k \left(\sum_{m+n=k} \frac{(-1)^m}{m!n!} \phi(z a^n, a^m) \right), \end{aligned}$$

since ϕ is a continuous bilinear map. Therefore

$$(1) \quad \sum_{k=1}^{\infty} t^k \left(\sum_{m+n=k} \frac{(-1)^m}{m!n!} \phi(z a^n, a^m) \right) = 0$$

for any $t \in \mathbb{R}$.

Let $\tau \in A^*$. We define a map $f : \mathbb{C} \rightarrow \mathbb{C}$ by

$$f(\lambda) = \tau \left(\sum_{k=1}^{\infty} \lambda^k \left(\sum_{m+n=k} \frac{(-1)^m}{m!n!} \phi(za^n, a^m) \right) \right)$$

for all $\lambda \in \mathbb{C}$. Hence from (1) for $t \in \mathbb{R}$, we find that $f(t) = 0$. So f is an analytic function on real axis and hence on $\mathbb{C}$ by Lemma 2.1. Now since f is zero for each point on a real axis, by Lemma 2.2, $f(\lambda)$ is identically equal to zero throughout $\mathbb{C}$ and τ arbitrary, consequently,

$$(2) \quad \sum_{m+n=k} \frac{(-1)^m}{m!n!} \phi(za^n, a^m) = 0$$

for all $a \in A_{sa}$ and $k \in \mathbb{N}$. Let $k = 1$, we find that $\phi(za, 1) - \phi(z, a) = 0$ and hence

$$(3) \quad \phi(za, 1) = \phi(z, a)$$

for all $a \in A_{sa}$. Now taking $k = 2$ in (2), we obtain $\frac{1}{2}\phi(za^2, 1) - \phi(za, a) + \frac{1}{2}\phi(z, a^2) = 0$ for any $a \in A_{sa}$. So by (3) we have

$$(4) \quad \phi(za, a) = \phi(za^2, 1), \quad \forall a \in A_{sa}.$$

For any $a, b \in A_{sa}$, replacing a by $a + b$ in (4), we get that

$$\phi(za, b) + \phi(zb, a) = \phi(z(ab + ba), 1).$$

If we define the linear map $\Phi : A \rightarrow X$ by $\Phi(a) = \phi(za, 1)$, then Φ is continuous and

$$\phi(za, b) + \phi(zb, a) = \Phi(a \circ b)$$

for all $a, b \in A_{sa}$. Since every element $a \in A$ is a combination of some self adjoints, we also deduce above statements are for each arbitrary elements of A . $\square$

Let $a \in A_{sa}$ and $t \in \mathbb{R}$. From $(\exp(ta)z^*, \exp(-ta)) \in S_A^{l*}(z)$, and using similar arguments as proof of above theorem we get the following result.

Theorem 3.2. *Let X be a Banach space and let $\phi : A \times A \rightarrow X$ be a continuous bilinear map with the property that*

$$\phi(a^*, b) = \phi(z, 1) \text{ for all } (a, b) \in S_A^{l*}(z).$$

Then

$$\phi(za, a) = \phi(za^2, 1) \text{ and } \phi(za, 1) = \phi(z, a), \quad a \in A$$

and there exists a continuous linear map $\Phi : A \rightarrow X$ such that

$$\phi(za, b) + \phi(zb, a) = \Phi(a \circ b), \quad a, b \in A.$$

Remark 3.3. Let $a \in A_{sa}$ and $t \in \mathbb{R}$. By the fact that $(\exp(ta), z^* \exp(-ta)) \in S_A^{r*}(z)$ and $(\exp(ta), \exp(-ta)z) \in S_A^{l*}(z)$, and using similar arguments as proof of Theorem 3.1 we get the following:

Let X be a Banach space and let $\phi : A \times A \rightarrow X$ be a continuous bilinear map. If ϕ satisfies any of the following conditions

- (i) $\phi(a, b^*) = \phi(1, z)$ for all $(a, b) \in S_A^{r*}(z)$;
- (ii) $\phi(a^*, b) = \phi(1, z)$ for all $(a, b) \in S_A^{l*}(z)$,

then

$$\phi(a, az) = \phi(1, a^2z) \text{ and } \phi(1, az) = \phi(a, z), \quad a \in A$$

and there exists a continuous linear map $\Phi : A \rightarrow X$ such that

$$\phi(a, bz) + \phi(b, az) = \Phi(a \circ b), \quad a, b \in A.$$

The results obtained are especially important in the case where the $z \neq 0$. We have the following corollaries.

Corollary 3.4. *Let X be a Banach space and let $\phi : A \times A \rightarrow X$ be a continuous bilinear map. If ϕ satisfies any of the following conditions;*

- (i) $a, b \in A, ab^* = 1 \Rightarrow \phi(a, b^*) = \phi(1, 1)$,
- (ii) $a, b \in A, a^*b = 1 \Rightarrow \phi(a^*, b) = \phi(1, 1)$,

then

$$\phi(a, b) + \phi(b, a) = \phi(a \circ b, 1)$$

for all $a, b \in A$.

Proof. The result is clear from Theorems 3.1 and 3.2 by letting $z = 1$. $\square$

Recall that a bilinear map $\phi : A \times A \rightarrow X$ is called symmetric if $\phi(a, b) = \phi(b, a)$ holds for all $a, b \in A$. By Theorem 3.1 and Corollary 3.4, the following corollary is obvious.

Corollary 3.5. *Let X be a Banach space and let $\phi : A \times A \rightarrow X$ be a continuous symmetric bilinear map. Then the following conditions are equivalent:*

- (i) $a, b \in A, ab^* = 1 \Rightarrow \phi(a, b^*) = \phi(1, 1)$;
- (ii) $a, b \in A, a^*b = 1 \Rightarrow \phi(a^*, b) = \phi(1, 1)$.

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